Computing Option Values

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Pricing Options

- If t = T, we know the value; it is the payoff.
- For t < T, we could decide the value of the option if we knew what the future evolution of the underlying stock price will be.
- We can not know for sure…
- … but we might be able to build a model of stock price evolution and use it to make reasonable predictions.

Evolution of Stock Prices

Stock Price Evolution Model

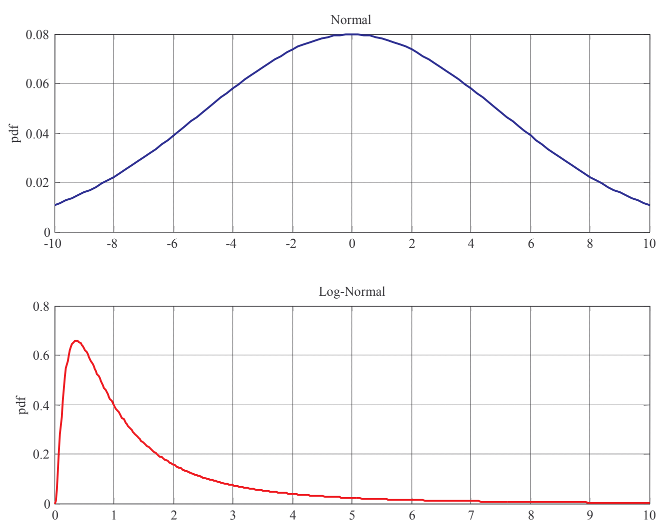
- We will model the evolution of prices in the interval [t,T]. We divide this into intervals of equal length Δ.

$$
\begin{aligned}
S(T) &= \left[\frac{S(T)}{S(T-\Delta)}\right]\left[\frac{S(T-\Delta)}{S(T-2\Delta)}\right]\cdots\left[\frac{S(t+\Delta)}{S(t)}\right]S(t) \\
0 &\leq \frac{S(t+(i+1)\Delta)}{S(t+i\Delta)} = e^{r(i\Delta)} \\
S(T) &= S(t)\exp(\sum_{i=0}^{N-1} r(i\Delta)) \\
Z(T) &= \sum_{i=0}^{N-1} r(i\Delta) = \log\frac{S(T)}{S(t)}
\end{aligned}
$$

Stock Price Evolution Model (2)

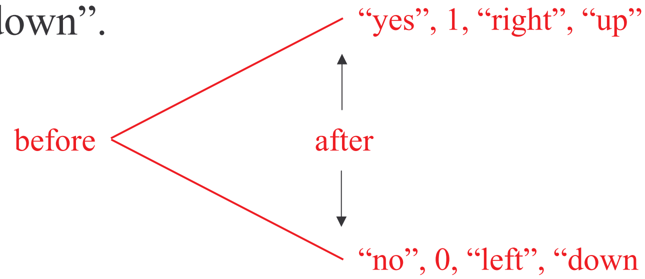
- Three assumptions:
  (1) $r(i\Delta)$ are i.i.d.;
  (2) $E[r(i\Delta)]=\mu\Delta$;
  (3) $var[r(i\Delta)]=\sigma^2\Delta$.
- Consequences:
  (1) $E[Z(T)] = \mu T$;
  (2) $var[Z(T)] = \sigma^2 T$.
  (3) Returns are normally distributed.
  (4) Prices are log-normally distributed.

Normal vs. Log-Normal



Bernoulli Random Variable

- Two possible outcomes, one has probability 0 < p < 1, the other has probability 1 – p.
- The outcomes could be interpreted as “yes” & “no”, 1 & 0, “left” & “right”, “up” & “down”.



Binomial Distribution

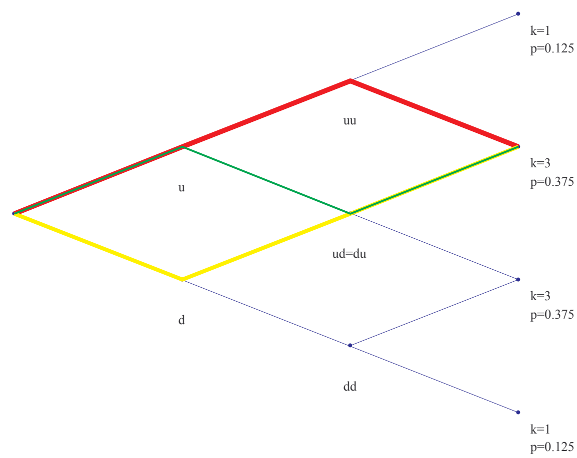
- Sum of **n** Bernoulli variables.
- Bernoulli variables: outcome is **1** with probability **p**, and **0** with probability **1-p**.
- Let **b** be a binomial variable.

$$p(b = k) = \binom{n}{k} p^k (1 - p)^{n-k}, 0 \leq k \leq n$$

$$E[b] = np$$

$$Var[b] = np(1 - p)$$

Binomial Distribution (2)



n = 3
p = 0.5 => symmetry!

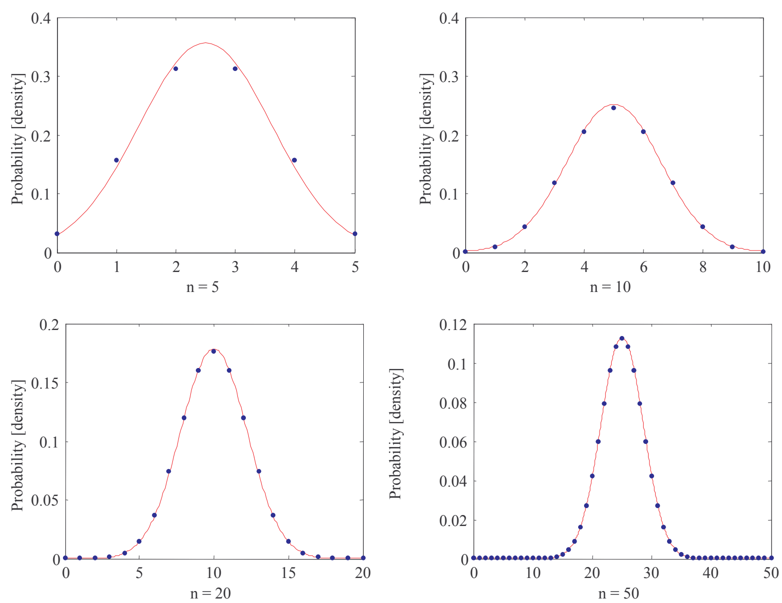
Lattices

- For given n, they contain $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$ nodes.
- They "forget the past," i.e. it does not matter how one matches a state. E.g. state **ud** is the same as state **du**. It does not matter how you got into state S, once there, the past does not influence the set of states you can reach from S.
- This is, in fact, not a problem for options. Why?

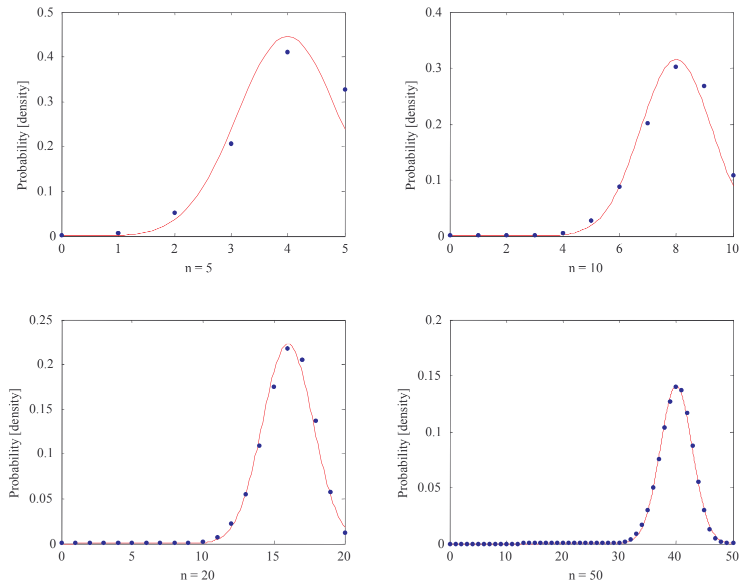
Binomial vs. Normal

- The mean (expectation) of a binomial variable is **np**, its variance is **np(1-p)**.
- A binomial variable with parameters **n** and **p**, assuming that **n** is large, can be approximated by, or it can be used to approximate, a normal distribution with mean **np** and variance **np(1-p)**.
- **B(n, p) ~ N(np, np(1-p))**

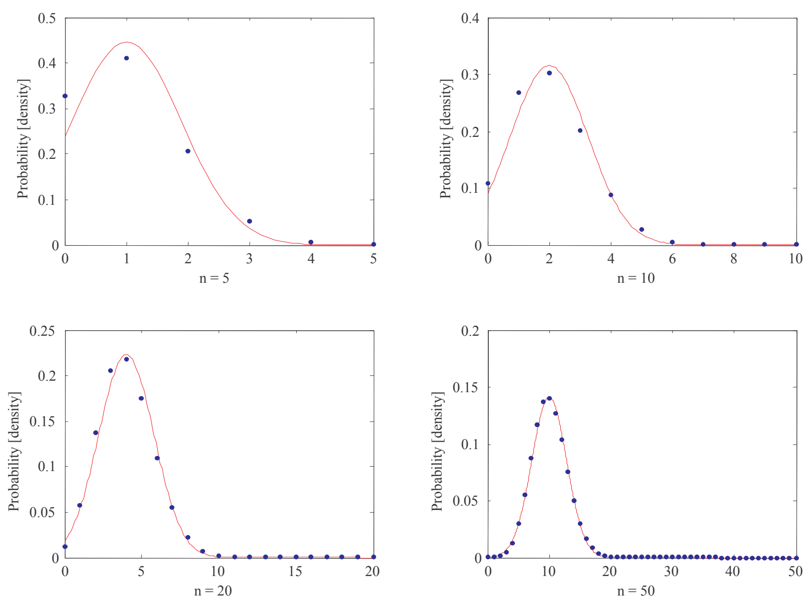
Binomial vs. Normal (p=0.5)



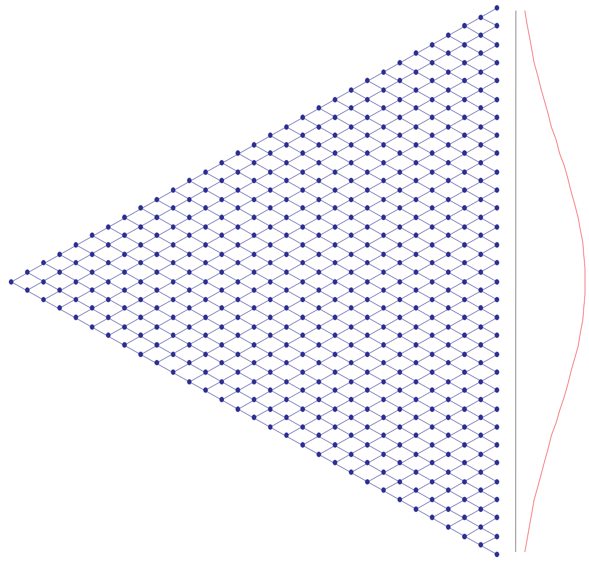
Binomial vs. Normal (p=0.8)



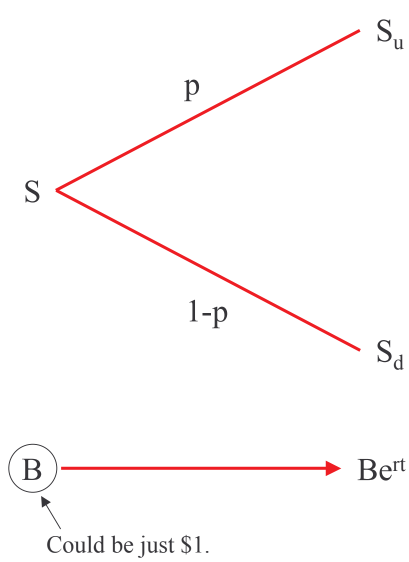
Binomial vs. Normal (p=0.2)



A Bigger Lattice (p = 0.5)



One-Period Binomial Model



- One initial state;
- Two final states "up" and "down";
- True (real-life) probability of "up" state occurring: $0<p<1$;
- Can think of p as "my estimate" of the probability;
- Two instruments: S and B;
- WLOG $S_u > S_d$;
- A continuously-compounded interest rate r is paid on B.

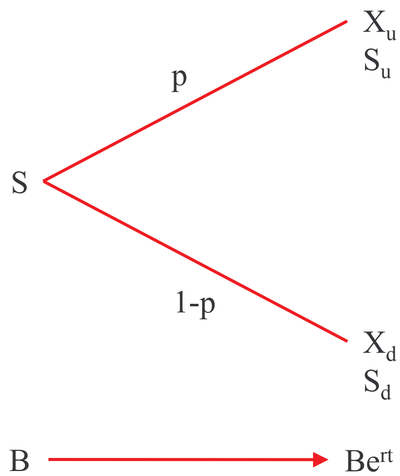
No Arbitrage

- If $e^{rt} > r_u = S_u/S$, then we can borrow S at t = 0, sell it, and invest the proceeds in B. At the end, we buy back the share, and we make either $Se^{rt} - S_u > 0$, or $Se^{rt} - S_d > S_u - S_d > 0$.
- What about $e^{rt} < r_d = S_d/S$?
- What about $e^{rt} = r_u = S_u/S$ or $e^{rt} = r_d = S_d/S$? (see next slide)
- To avoid arbitrage, we must have that **$S_d/S < r < S_u/S$**, i.e. the return on B must dominate r_d, and must be dominated by r_u.

No Arbitrage (2)

- What about $e^{rt} = r_u = S_u/S$? Let us apply the same strategy as if we had $e^{rt} > r_u = S_u/S$. The payoff at t is $Se^{rt} - S_u = 0$, or $Se^{rt} - S_d = S_u - S_d > 0$. So we make money with probability p. We are not guaranteed to make money in any period of time, but – on average – we will get a payoff of $(S_u - S_d)p > 0$, with **no risk of loss**! This would still be arbitrage.
- The case of $e^{rt} = r_d = S_d/S$ is treated similarly.

Reproducing Payoffs



- Consider an arbitrary payoff X, which depends on the final state.
- We will determine a portfolio of S and B that reproduces X.
- We need assumptions…

Reproducing Payoffs (2)

$$\begin{cases} n_s S_u + n_B B e^{rt} = X_u \\ n_s S_d + n_B B e^{rt} = X_d \end{cases}$$

$$\begin{cases} n_s = \frac{X_u - X_d}{S_u - S_d} \\ n_B = \frac{1}{Be^{rt}} \frac{S_u X_d - S_d X_u}{S_u - S_d} = \frac{1}{Be^{rt}} \left(X_u - \frac{X_u - X_d}{S_u - S_d} S_u \right) = \frac{1}{Be^{rt}} (X_u - n_s S_u) \end{cases}$$

The value at time 0 of the portfolio which reproduces the payoff X at time t is:

$$V = n_s S + n_B B = n_s S + \frac{1}{e^{rt}} (X_u - n_s S_u)$$

What if the market trades this portfolio at a different price?

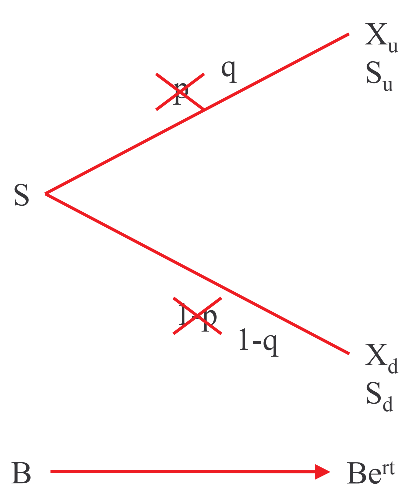
Reproducing Payoffs (3)

$$
\begin{aligned}
V &= e^{-rt}\left[\frac{Se^{rt}-S_d}{S_u-S_d}X_u+\frac{S_u-Se^{rt}}{S_u-S_d}X_d\right] \\
&= e^{-rt}\left[\frac{Se^{rt}-S_d}{S_u-S_d}X_u+\frac{S_u-S_d-Se^{rt}+S_d}{S_u-S_d}X_d\right] \\
&= e^{-rt}\left[\frac{Se^{rt}-S_d}{S_u-S_d}X_u+\left(1-\frac{Se^{rt}-S_d}{S_u-S_d}\right)X_d\right] \\
&= e^{-rt}\left[qX_u+(1-q)X_d\right]
\end{aligned}
$$

$$
0 < q = \frac{Se^{rt}-S_d}{S_u-S_d} = \frac{e^{rt}-\frac{S_d}{S}}{\frac{S_u}{S}-\frac{S_d}{S}} < 1
$$

Looks like a probability… it is (can be considered) a probability!

Reproducing Payoffs (4)



- **q** is independent of (our estimate of) the true probability **p**.
- We will call q … equivalent martingale probability.
- **p** is irrelevant.
- We will agree on q, as long as we agree on the states.

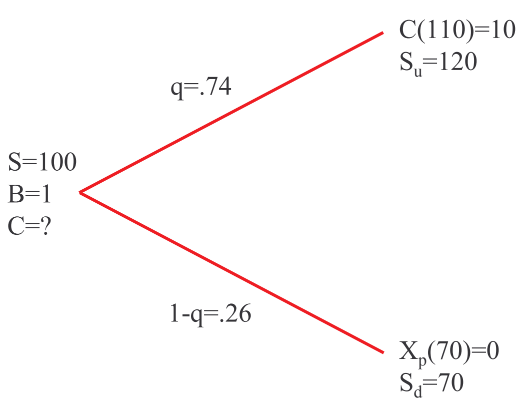
Risk-Neutral Valuation

- The time 0 value of the portfolio that reproduces an **arbitrary** payoff at time t:

$$V = e^{-rt}\left[qX_u + (1-q)\,X_d\right]$$

- q,1-q = equivalent [martingale] probabilities
- We do not need to know, nor do we care about the true probability **p**.
- If arbitrage is possible, this reasoning does not hold.

Pricing a Call



$$e^{rt} = 1.07$$

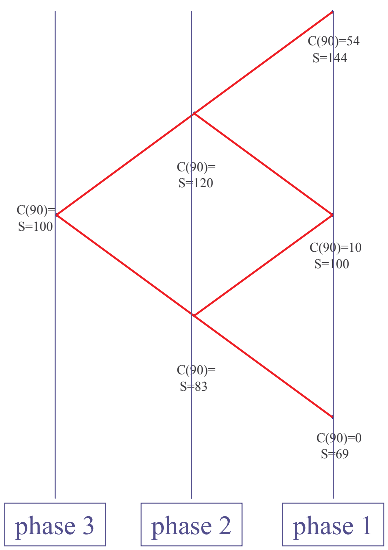
$$\begin{aligned} q &= \frac{100 \cdot 1.07 - 70}{120 - 70} = 0.74 \\ V &= \frac{1}{1.07}(.74 \cdot 10 + (1 - .74) \cdot 0) = 6.92 \end{aligned}$$

$$\begin{cases} 120 n_s + 1.07 n_B = 10 \\ 70 n_s + 1.07 n_B = 0 \end{cases}$$

$$\begin{cases} n_s = .20 \\ n_B = -13.08 \end{cases}$$

$$V = 100 \cdot .20 - 13.08 = 6.92$$

Multiple Period Call



- Start at the “deep” end of the lattice.
- Work backwards, one step at a time.
- Get the final value at the root (initial state).
- This idea can be generalized to an arbitrary number of levels.
- Would you treat an American call differently from an European call? How? Why?
- This lattice does not “remember” the past. Can you suggest a modification that eliminates this problem?